

Quantile Estimation As a Tool for Calculating VaR

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Abstract: Two cases are observed and their corresponding calculations for getting the VaR is shown. At first we make the assumption that the return of the portfolio is normal. Then we change our supposition and assume that the return of a portfolio is lognormal.

Keywords: Value-At-Risk, Normal distribution, Lognormal distribution, quantile

1. Introduction

VaR (Value-at-Risk) summarizes the expected maximum loss over a target horizon with a given level of confidence. VaR answers the question: how much can I lose with x% probability over a given time horizon. It captures risk in a single number!

If a portfolio manager has calculated a VaR of 1 Million Dollar assuming a confidence level of 99%, this VaR tells him that he can expect having only one loss exceeding 1 Million Dollar out of 100 days.

Normally in an enterprise-wide risk model you will find a confidence level of 95%. Defining this problem in mathematical terms you have to find the solution of

$$\int_{-\infty}^{VaR} f_{P/L}(l) dl = 0.05$$

where we denoted by $f_{P/L}(\cdot)$ the profit/loss function.

2. A simple example using the normal distribution

Assume that a portfolio manager manages a portfolio **P** which consists of a single asset XXX. **P** has a market value of 100 Million Dollar. We assume that the return of the portfolio is normally distributed with annual mean return of 10% and annual standard deviation of 30%.

Therefore the value of the portfolio **P** has the distribution:

$$(\text{Values of P}) \sim \mathcal{N}(110;900)$$

(NV stands for normal distribution).

Now we want to answer the question what the likelihood is that **P** has the value of 80 Million Dollars (or less) at the end of our time horizon. That means, we have to define a random variable (in the following: r.v.) ${}^{1|0}\mathbf{P}$ which can take on the possible values of the portfolio. Now we calculate:

$$\Pr({}^{1|0}\mathbf{P} \leq 80), {}^{1|0}\mathbf{P} \sim \mathcal{N}(110 ;900) \quad [2.1]$$

We assume: $Z \sim \mathcal{N}(0;1)$. Then the following holds:

$$\Pr({}^{1|0}\mathbf{P} \leq 80) = \Pr(30Z + 110 \leq 80) \quad [2.2]$$

$$= \Pr(Z \leq (80-110) / 30) \quad [2.3]$$

$$= \Pr(Z \leq -1) \quad [2.4]$$

$$= \Phi_Z(-1) \quad [2.5]$$

$$= 0,1587 \quad [2.6]$$

The probability that our portfolio **P** has a value of 80 Million Dollars or less at the end of our time horizon is approximately 15,9 %.

Now we want to find out what loss can occur by a confidence level of 99 %. That means, we have to search for the ,01-quantile $\Phi^{-1}(,01)$ of the r.v. ${}^1|_0\mathbf{P}$.

We compute:

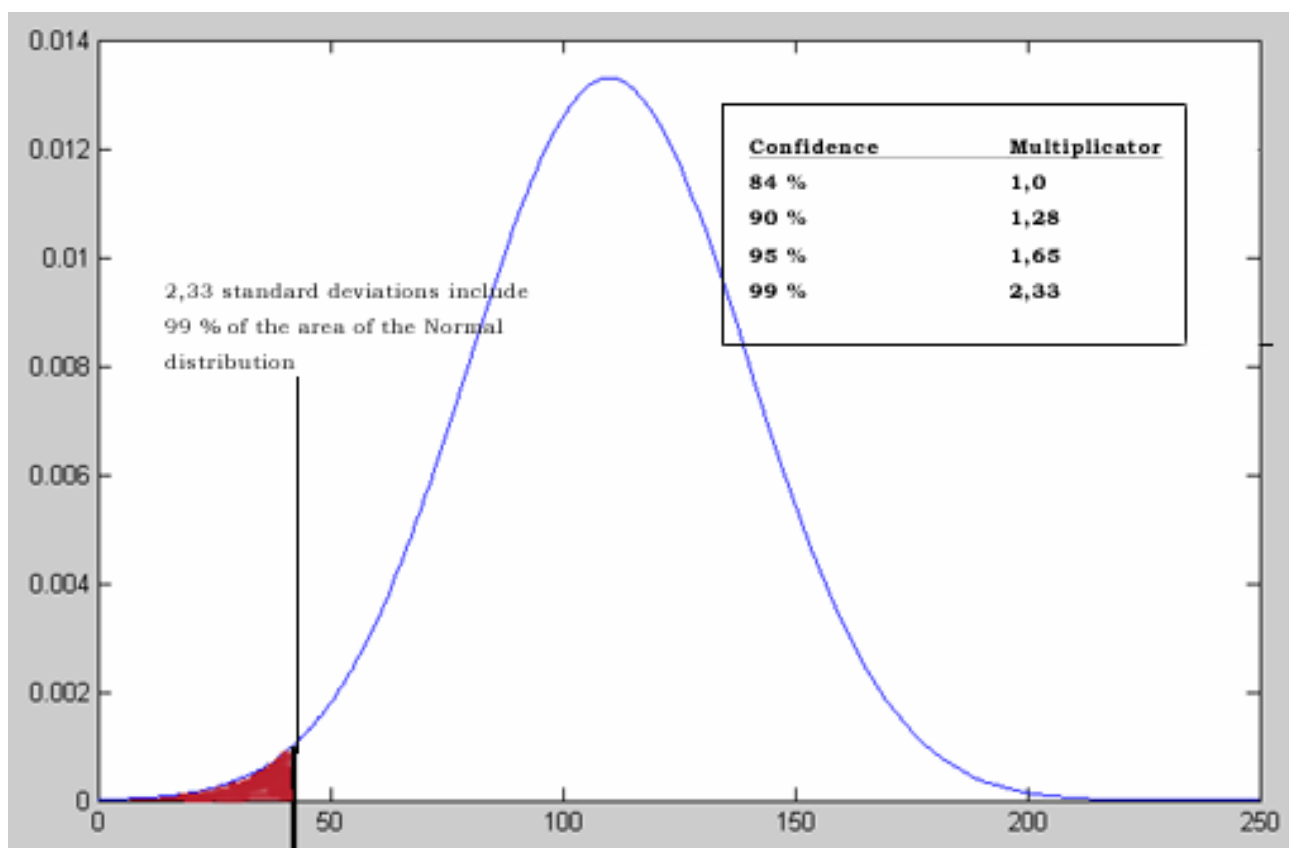
$$\Phi^{-1}(,01) = 110 - 2,33 * 30 \quad [2.7]$$

$$= 40,1 \quad [2.8]$$

using $\Phi(2.33) = 0.99$

With the probability of 1 % our portfolio \mathbf{P} has only a value of 40,1 Million Dollars at the end of our time horizon. The VaR in this case has the value of $100 - 40,1 = 59,9$ million Dollar.

Figure 1: Normal distribution



We can see that 1 % of the area under the density function is at the left to the mark (the dark fill). We can say: The possible values of the r.v. ${}^1|_0\mathbf{P}$ are normally distributed and with the probability of 1 % the value of ${}^1|_0\mathbf{P}$ lies in the dark zone.

3. The lognormal distribution

Frequently, especially during the pricing of assets the choice that the returns follow a lognormal distribution is more reasonable than the choice for a normal distribution (for example: a lognormally distributed r.v.'s can't take on negative values¹). Considering this we update our assumption [2.1] to

$$\Pr({}^1|_0\mathbf{P} \leq 80), {}^1|_0\mathbf{P} \sim \mathcal{L}\mathcal{O}\mathcal{G}(m; s^2), \quad [3.1]$$

where

$$m = \ln\left(\frac{\mu^2}{\sqrt{\sigma^2 + \mu^2}}\right)$$

$$s = \sqrt{\ln\left(\left[\frac{\sigma}{\mu}\right]^2 + 1\right)}.$$

With $\mu = 110$ und $\sigma^2 = 900$ we get

$$m = 4,66, s = 0,26$$

We carry out a similar calculation [2.2] to [2.6]:

$$\Pr({}^1|_0\mathbf{P} \leq 80) = \Pr(\ln({}^1|_0\mathbf{P}) \leq \ln(80)) \quad [3.2]$$

$$= \Pr(0,26Z + 4,66 \leq \ln(80)) \quad [3.3]$$

$$= \Pr(Z \leq ((\ln(80) - 4,66) / 0,26)) \quad [3.4]$$

$$= \Phi_Z(-1,07) \quad [3.5]$$

$$= 0,1432. \quad [3.6]$$

¹ Remember: $X \sim \mathcal{N}(\mu, \sigma^2) \implies \exp(X) \sim \mathcal{L}\mathcal{O}\mathcal{G}(\mu, \sigma^2)$

For the ,01-quantile $\Phi^{-1}(,01)$ we get:

$$\Phi_Z^{-1}(-2,33) = 0,01$$

$$\Rightarrow \Pr(Z \leq -2,33) = 0,01 \quad [3.7]$$

$$\Rightarrow \Pr([\ln({}^1|{}^0\mathbf{P}) - m] / s \leq -2,33) = 0,01 \quad [3.8]$$

$$\Rightarrow \Pr([\ln({}^1|{}^0\mathbf{P}) - 4,66] / 0,26 \leq -2,33) = 0,01 \quad [3.9]$$

$$\Rightarrow \Pr(\ln({}^1|{}^0\mathbf{P}) \leq 4,66 - 2,33(0,26)) = 0,01 \quad [3.10]$$

$$\Rightarrow \Pr({}^1|{}^0\mathbf{P} \leq 1,4) = 0,01 \quad [3.11]$$

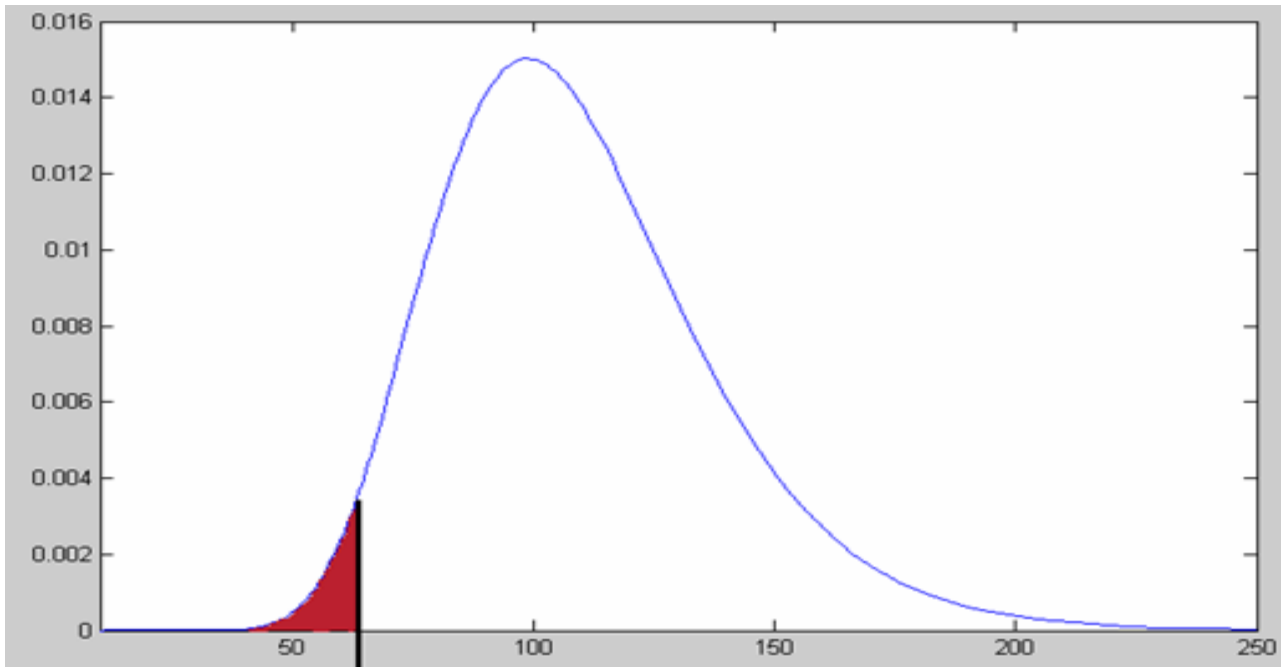
We deduct:

$$\Phi^{-1}(0,01) = 110 - 1,4 * 30$$

$$= 68.$$

The assumption that ${}^1|{}^0\mathbf{P}$ is lognormally distributed lead us to the following results: With a probability of 14,3 % the portfolio \mathbf{P} will have a value of 80 Million dollar (or less) at the end of our time horizon. And with a probability of 1 % ${}^1|{}^0\mathbf{P}$ has only a value of 68 Million dollars at the end of our time horizon. That means that VaR has a value of $(100 - 68 =) 32$ Million dollar. Again we can explain the results by looking at the lognormal distribution:

Figure 2: Lognormal distribution



At the left to the mark we find the values of the portfolio ${}^{1|0}\mathbf{P}$ which will show up with a probability of 1 % or less.